1 Partitions and Cartesian products

1.1 Partitions

Let us recall what a partition is:

Definition 1. *A partition of a set U is a collection of nonempty subsets of U which are pairwise disjoint and whose union is the entire set U.*

As we have seen in Example 1.8 in your textbook, we had

- $X = \{2, 4, 6, 8, 10, 11, 12, 13, 14, 16, 18\}$,
- $X_1 = \{2, 4, 6, 8\}$,
- $X_2 = \{11, 13\}$,
- $X_3 = \{10, 12, 14, 16, 18\}$.

Then X_1, X_2 and X_3 form a partition of X, because X is exactly the same as $X_1 \cup X_2 \cup X_3$, and X_1, X_2 and X_3 are pairwise disjoint.

Moreover, we saw the following: $n(X) = 11$, $n(X_1) = 4$, $n(X_2) = 2$, $n(X_3) = 5$. Notice that $n(X) = n(X_1) + n(X_2) + n(X_3)$. The reason why we have this formula is because we have a partition for X.

In general, if $X_1, X_2, \ldots, X_k$ form a partition of X, then we also have the formula

$$n(X) = n(X_1) + n(X_2) + \cdots + n(X_k).$$

This is called the partition principle.

1.2 Cartesian product

Now we go back to Cartesian product. Recall that the Cartesian product of sets A and B is defined to be the set

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

In words, $A \times B$ contains all the ordered pairs (a, b) with $a \in A$ and $b \in B$. We are also interested in the size of $A \times B$ (that is, we would like to know $n(A \times B)$). But how do we find that?

Recall that in Example 1.5 (in Section 1.1), we saw that if $A = \{a, c, e\}$ and $C = \{b, d\}$, then

$$A \times C = \{(a, b), (a, d), (c, b), (c, d), (e, b), (e, d)\}.$$

Notice that $n(A \times C) = 6$. Also, $n(A) = 3$ and $n(C) = 2$. In particular, we see that $n(A \times C) = n(A) \cdot n(C)$. In fact, in general, for *any* sets S and T, we have

$$n(S \times T) = n(S) \cdot n(T).$$

We can also find the size of Cartesian product of more than two sets. For example, if E, F, G are sets, then their Cartesian product $E \times F \times G$ contains all ordered triples (e, f, g) with $e \in E$, $f \in F$ and $g \in G$. The corresponding general result is as follows:

If $Y_1, Y_2, \ldots, Y_k$ are sets, then

$$n(Y_1 \times Y_2 \times \cdots \times Y_k) = n(Y_1) \cdot n(Y_2) \cdots n(Y_k).$$

How is it different from the partition principle? Well, in one formula we have sum, while in the other one we have product instead. Second, the formula for partition principle holds *only* for partitions, but the one for Cartesian product holds for *any* sets.

In the class, I talked about why the formula for the size of Cartesian product should hold, using partition. If you are not able to understand that completely, that would be fine. It is more important to know when to use these formulas.

2 More examples

Here I give two more examples, which might be helpful.

1. Let A and B be disjoint subsets in a universal set U with $n(U) = 50$, $n(A \cup B) = 35$ and $n(B') = 25$. Find $n(A)$.

   Solution: Note that A and B are disjoint, and hence by the partition principle, $n(A \cup B) = n(A) + n(B)$. So $n(A) + n(B) = 35$.

   As we want to find $n(A)$, we need to know $n(B)$. How to find that? Well, observe that B, B' actually form a partition of U. Therefore, again by the partition principle, we have $n(U) = n(B) + n(B')$. So $50 = n(B) + 25$, which yields $n(B) = 25$.

   Therefore, back to $n(A) + n(B) = 35$, now we have $n(A) + 25 = 35$, which gives $n(A) = 10$.

2. A set X with $n(X) = 45$ is partitioned into subsets X_1, X_2 and X_3. If $n(X_2) = 2n(X_1)$, $n(X_3) = 3n(X_2)$, find $n(X_1)$.

   Solution: In this type of problems, it is usually convenient letting $x = n(X_1)$ and then find what x is.

   Let $x = n(X_1)$. Then we have $n(X_2) = 2n(X_1) = 2x$ and $n(X_3) = 3n(X_2) = 3(2x) = 6x$.

   By the partition principle, we have $n(X) = n(X_1) + n(X_2) + n(X_3)$. That is, $45 = x + 2x + 6x$, which yields $9x = 45$ and hence $x = 5$. So we have $n(X_1) = 5$.